Robust Unconstrained Predictive Control Design with Guaranteed Nominal Performance

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A systematic and complete method to design robust predictive controllers for unconstrained linear systems is proposed. The synthesis procedure is based on rigorous theoretical foundations, without resorting to approximations or ad hoc design guidelines, yet it remains a viable tool for practical application. A significant feature is that the robust predictive controller retains the servo performance of a nominal predictive controller designed using conventional methods. In addition, the robust predictive controller can be designed to guarantee perfect steady-state rejection of asymptotically constant disturbances. The robust design method is developed for systems affected by unmodeled dynamics and is based on solving a discrete-time model-matching problem. It is shown that the robust controller can be classified legitimately as a predictive controller because it minimizes the same performance functional as the nominal predictive controller. An illustrative design example is given.

Introduction

Predictive control strategies have received much attention in the literature and have been accepted in industry. The popularity of these methods is due to the fact that they offer good performance, are easy to understand and formulate, and can accommodate input/output process constraints. The industrial success of the predictive-control techniques is attested to by the variety of commercial predictive controllers that are available to the chemical processing industry through specialized vendors. In fact, predictive control is possibly the most widely utilized model-based control strategy in the chemical industry, and often the only model-based control technique supported and offered commercially by control engineering companies. Seborg (1994) reports that in oil refineries and petrochemical plants around the world there are hundreds, perhaps thousands, of predictive controllers deployed.

It is possible to design predictive controllers using different plant representations. One common approach is to use finite impulse-response models (FIR). This is the favored approach in the earlier formulations, including dynamic matrix control (DMC) (Cutler and Ramaker, 1980), model algorithmic control (Mehra et al., 1979), and the quadratic DMC formulation of Garcia and Morshedi (1986). An extensive review of FIR-based predictive control is given in Garcia et al. (1989). The FIR methods are applicable only to stable plants, and often require large model orders, typically retaining 30 to 40 impulse response coefficients. An alternative approach is to base the controller design on transfer-function models, which are applicable to both stable and unstable plants, and lead to lower-order representations. Examples of transfer-function-based predictive control are the well-known generalized predictive control (GPC) technique (Clarke et al., 1987) and the MUSMAR approach (Greco et al., 1984). A third approach is the use of state-space methods for predictive control design, a practice that has an early representative in the work of Kwon and Pearson (1977), and that has recently gained popularity through multiple advocates, such as the work of Muske and Rawlings (1993a,b). This large body of literature constitutes a rich source of knowledge to support the design and analysis of predictive controllers.

Currently, there is an increasingly visible interest in the research community in revisiting the predictive control design techniques with the intention of including robustness features that guarantee stability or adequate performance when the plant model is uncertain. One interesting example is the robust quadratic DMC design including hard constraints studied by Zafiriou (1990). The author uses a contraction mapping first proposed by Economou (1985) to successfully derive time-domain conditions for robust stability with respect to uncertainty in the impulse-response coefficients of the nominal model. Unfortunately, this approach may lead to conservative designs, and is not of practical utility because it involves a very high numerical computation effort. More recently, Genceli and Nikolaou (1993) proposed an analysis and synthesis method for predictive controllers based on FIR models, including constraints and using a linear cost functional instead of the classic quadratic functional. These authors use a parametric model uncertainty description that bounds the maximum deviations allowed for each pulse-response coefficient, and obtain a sufficient condition for robust closed-loop stability.

The robustness of predictive controllers designed using transfer-function representations is receiving increasing attention in the literature. Kouvaritakis et al. (1992) propose an alternative approach to GPC that employs an additional compensator that stabilizes the plant before the predictive design is carried out. These authors make use of the Q-parametrization procedure popularized by Youla et al. (1976) as the basis for a scheme that makes the controller robust with respect to unstructured perturbations. The authors state rigorous necessary and sufficient conditions for robust stability; however, the approach proposed for synthesizing robust controllers is an approximate, albeit practical scheme. The method consists of using polynomial or fixed-order transfer-function approximations for the Youla parameter, and least-squares methods to identify the parameters of the robust design. Robinson and Clarke (1991) analyzed the robustness of the GPC technique in an indirect fashion. They investigated two particular control designs, namely, a dead-beat and a mean-level controller, which can be interpreted as special cases of GPC that arise for specific tuning parameters. The focus is on the effect of a polynomial prefilter T proposed by the authors to introduce stability robustness. The analysis is not strictly valid for any other choices of predictive-control tuning parameters, nor for the case of unstable plants. In a recent publication, Yoon and Clarke (1995) compare designs based on T-filtering and on Q-parametrization, and propose simple guidelines for robust synthesis using the T polynomials. A representative approach to robust control of constrained nonlinear systems in continuous time is reported in Michalska and Mayne (1993).

This article presents a systematic procedure for making predictive controllers robust in the presence of unmodeled dynamics. A *nominal predictive controller* designed using the nominal plant model is parametrized to produce a *robust predictive controller* that ensures stability with respect to the uncertainty in the nominal model, and guarantees adequate performance. Given that so much knowledge is currently available for designing and tuning high-performance predictive controllers for the case where the plant model is assumed to be free of uncertainty, it is of fundamental importance that the robustified controller be able to preserve the performance of the nominal controller. Specifically, in the limit when the uncertainty in the plant model is negligible, the servo performance of the robust predictive controller should closely resemble the tracking behavior of the nominal predictive controller. Furthermore, in its role as a regulator the robustified predictive controller must be able to reject the effect of asymptotically constant perturbations.

The proposed approach consists of parametrizing a nominal predictive controller that is designed using conventional and well-established methods. A significant feature is the use of a Q-parametrization technique that preserves the servo dynamics of the nominal controller. The method is applicable to unconstrained predictive control designs that use transfer-function plant models corrupted by unstructured uncertainty. A solution to the robust design problem is obtained using an algorithm by Rotstein and Sideris (1992) that yields an explicit solution to an underlying Nehari extension problem. Therefore, the technique avoids the use of approximations yet remains practical. The design also has the advantage of being able to include an integrator in the robust controller, thus enhancing the closed-loop performance properties by guaranteeing steady-state rejection of asymptotically constant disturbances.

The article is organized as follows. The next section presents a concise review of the design equations for a nominal predictive controller, including the resulting control law in transfer-function form. The third section derives a necessary and sufficient condition for robust stability, and the fourth section develops a comprehensive methodology for synthesizing robust predictive controllers. A design example is given in the fifth section, followed by final conclusions offered in the last section.

Nominal Predictive Control Design

The design of nominal predictive controllers is well-documented in the literature. In particular, a wealth of knowledge is available to resolve crucial design issues such as nominal closed-loop stability, and parameter tuning (Lambert, 1987; Mohtadi, 1987). Typically, predictive controllers are deployed by executing at every sampling instant an algorithm that solves a quadratic optimization problem. For analysis purposes, it is desirable to represent the algorithmic controller in terms of transfer functions, thus allowing the utilization of classic z-domain tools for analyzing stability and performance. This section presents a brief review of the analysis technique discussed in Crisalle et al. (1989), which casts an algorithmic predictive-control law of the GPC type into a form involving transfer-function operators. The resulting nominal controller is used later as the basis for the design of a robust controller.

Consider the nominal process model

$$A(z)y(z) = B(z)u(z), \tag{1}$$

where $y(z)$ and $u(z)$ are the process output and input, respectively, and $A(z)$ and $B(z)$ are the coprime polynomials

$$A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \tag{2}$$

$$B(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0 \tag{3}$$

of order n and m, respectively, where $n > m$. Predictive control involves the minimization of the quadratic cost functional

$$J(t) = \sum_{i=1}^{N_y} [r(t+i) - y(t+i|t)]^2 + \lambda \sum_{i=1}^{N_u} [\Delta u(t+i)]^2, \tag{4}$$

where $\{r(t+i)\}$ is the sequence of future values of the setpoint, $\{y(t+i|t)\}$ is the sequence of predicted future values of the output, $\{\Delta u(t+i)\}$ is the sequence of future control increments, λ is the move-suppression parameter used to penalize excessive control energy, and parameters N_y and N_u are the prediction and control horizons, respectively. By definition, a predictive control law is an algorithm that at every sampling instant produces the control move $u(t)$ that minimizes the functional, Eq. 4, for the prescribed setpoint sequence $\{r(t)\}$. The optimal control move is naturally found by differentiating Eq. 4 with respect to the control moves, equating the result to zero, and solving for $u(t)$. Following the development in Crisalle et al. (1989) it is possible to write the resulting control law in terms of transfer-function operators in the form

$$\frac{R(z)}{z^n}u(z) = T(z)r(z) - \frac{S(z)}{z^n}y(z), \tag{5}$$

which includes the polynomial operators

$$R(z) = z^n + r_{n-1}z^{n-1} + \cdots + r_o \tag{6}$$

$$S(z) = s_n z^n + s_{n-1}z^{n-1} + \cdots + s_0 \tag{7}$$

$$T(z) = t_{N_y}z^{N_y} + t_{N_y-1}z^{N_y-1} + \cdots + t_1 z \tag{8}$$

where

$$R(1) = 0, \tag{9}$$

$$T(1) = S(1), \tag{10}$$

and the coefficients of the moving-average polynomial $S(z)$, the regressor polynomial $R(z)$, and the setpoint advancement polynomial $T(z)$ are functions of the tuning parameters N_y, N_u, and λ, and of the model polynomials $A(z)$ and $B(z)$. Note that Eq. 9 implies that the predictive control law, Eq. 5, includes an integrator. A block-diagram representation of the predictive control structure is shown in Figure 1a. Specific design equations for the polynomials 6–8 are given in the Appendix; further details of the derivation can be found in Crisalle et al. (1989). A formulation equivalent to Eq. 5 is also derived in McIntosh et al. (1991).

Note that the transfer functions operating on $u(z)$ and $y(z)$ in the nominal predictive controller, Eq. 5, are of order n, the order of the nominal plant model. It is also significant to note that the setpoint advancement polynomial $T(z)$ is of degree equal to the prediction horizon N_y. Since $N_y \geq n$ is a common tuning prescription (Clarke et al., 1987), the order of $T(z)$ may exceed the order of $R(z)$, making the control law nonproper (noncausal) with respect to the setpoint signal. This noncausality is a natural consequence of the inclusion of future values of the setpoint in Eq. 4. Figure 1a shows that $T(z)$ acts on the setpoint to produce the intermediate signal $w(z) = T(z)r(z)$, which has the simple time-domain representation

$$w(t) = t_{N_y}r(t+N_y) + t_{N_y-1}r(t+N_y-1) + \cdots + t_1 r(t+1). \tag{11}$$

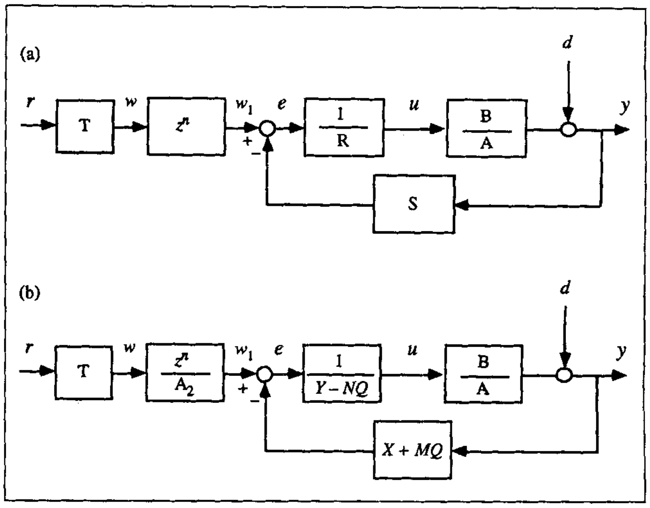

Figure 1. (a) Structure of a nominal predictive controller; (b) structure of the parametrized predictive controller featuring the Youla parameter $Q(z)$.

It is useful to remark that the nominal model, Eq. 1, and the functional, Eq. 4, are simpler versions of more elaborate formulations that improve the design performance at the expense of added complexity. Typical enhancements are the inclusion of a lower prediction-horizon parameter (Clarke et al., 1987), the inclusion of a weighted end-point term in Eq. 4 to guarantee stability for arbitrary parameter choices (Kwon et al., 1989; Demircioglu and Clarke, 1993), and the use of an auxiliary (filtered) setpoint. Another common practice is the addition of an exogenous stochastic input to Eq. 1; however, the inclusion of residual stochastic dynamics in the model is of less significance in the context of robust control design because such dynamics may conceivably be encompassed within the uncertainty model. These and other typical design choices can be accommodated within the framework proposed in this article through obvious modifications.

Figure 1a illustrates the closed-loop established when the nominal predictive controller, Eq. 5 is connected to the process, Eq. 1. In addition to the setpoint signal $r(t)$, the figure also shows an additive output-disturbance signal $d(t)$. Note that the servo dynamics of the closed loop are fully characterized by the equations

$$[A(z)R(z) + B(z)S(z)]y(z) = z^n B(z)T(z)r(z) \tag{12}$$

$$[A(z)R(z) + B(z)S(z)]u(z) = z^n A(z)T(z)r(z). \tag{13}$$

Therefore, the stability of the closed loop for a given nominal predictive controller can be easily established by calculating the roots of the characteristic polynomial $A(z)R(z) + B(z)S(z)$. Furthermore, due to the presence of the integral action, Eq. 9, in the controller and to the gain equality, Eq. 10, the closed-loop dynamics described by Eq. 12 are guaranteed to realize zero offset in the servo response. The integrator also guarantees perfect steady-state disturbance rejection for all disturbance signals that reach a constant steady state. These desirable performance characteristics of the nominal controller will be preserved in the robustified predictive controller designed in the following sections.

Controller Parametrization

In this section the nominal predictive controller, Eq. 5 is parametrized in terms of a transfer function $Q(z)$ selected in the spirit of the Wiener-Hopf design (Youla et al., 1976). However, a modification in the parametrization is introduced to achieve two important design requirements: (1) the parametrized controller must preserve the servo performance and the steady-state disturbance rejection properties of the nominal controller, and (2) the parametrized controller must also be a predictive controller.

Consider a nominal predictive controller, Eq. 5, that stabilizes the closed loop system, Eqs. 12 and 13. Because of the stability condition, the nominal closed-loop characteristic polynomial

$$A^*(z) = A(z)\mathrm{R}(z) + \mathrm{B}(z)\mathrm{S}(z) \tag{14}$$

of degree $2n$ is Schur. To parametrize the controller, consider a coprime fractional representation of the nominal plant model, Eq. 1, of the form

$$g_0(z) = \frac{\mathrm{B}(z)}{\mathrm{A}(z)} = \frac{N(z)}{M(z)}, \tag{15}$$

where $N(z)$ and $M(z)$ are proper and stable transfer functions that satisfy the Diophantine equation

$$N(z)X(z) + M(z)Y(z) = 1 \tag{16}$$

for some pair of stable and proper transfer functions $X(z)$ and $Y(z)$. (Note the use of italicized capital letters for transfer functions, while polynomials are designated with roman capital letters.) A suitable $[M(z), N(z)]$ pair can be readily derived from the nominal characteristic polynomial 14. The procedure consists of first factoring the closed-loop characteristic polynomial in the form $\mathrm{A}^*(z) = \mathrm{A}_1(z)\mathrm{A}_2(z)$, where both $\mathrm{A}_1(z)$ and $\mathrm{A}_2(z)$ are of degree n. If $\mathrm{A}^*(z)$ contains complex poles, then $\mathrm{A}_1(z)$ and $\mathrm{A}_2(z)$ are constructed such that each complex-conjugate pair is contained in either $\mathrm{A}_1(z)$ or $\mathrm{A}_2(z)$ to ensure that each polynomial factor has only real coefficients. Divide both sides of Eq. 14 by the factored characteristic polynomial to obtain

$$\frac{\mathrm{A}(z)\mathrm{R}(z)}{\mathrm{A}_1(z)\mathrm{A}_2(z)} + \frac{\mathrm{B}(z)\mathrm{S}(z)}{\mathrm{A}_1(z)\mathrm{A}_2(z)} = 1. \tag{17}$$

Finally, stable and proper factorizations that satisfy Eq. 16 are easily obtained by defining

$$M(z) := \frac{\mathrm{A}(z)}{\mathrm{A}_1(z)}, \qquad N(z) := \frac{\mathrm{B}(z)}{\mathrm{A}_1(z)} \tag{18}$$

and

$$X(z) := \frac{\mathrm{S}(z)}{\mathrm{A}_2(z)}, \qquad Y(z) := \frac{\mathrm{R}(z)}{\mathrm{A}_2(z)}, \tag{19}$$

where $X(z)$ and $Y(z)$ are clearly stable and proper rational transfer functions. This result allows us to write the nominal predictive control law, Eq. 5, in the equivalent form

$$Y(z)u(z) = Z(z)r(z) - X(z)y(z), \tag{20}$$

where

$$Z(z) := \frac{z^n}{\mathrm{A}_2(z)}\mathrm{T}(z). \tag{21}$$

The set of all solutions to Eq. 16 can be written in terms of the transfer functions 18 and 19, and a proper and stable transfer function $Q(z)$ through the well-known relations (Youla et al., 1976)

$$X'(z) = X(z) + M(z)Q(z) \tag{22}$$

$$Y'(z) = Y(z) - N(z)Q(z). \tag{23}$$

Therefore, the set of all stabilizing controllers with the structure, Eq. 20, is parametrized in the form

$$[Y(z) - N(z)Q(z)]u(z) = Z(z)r(z) - [X(z) + M(z)Q(z)]y(z) \tag{24}$$

to yield the control structure shown in Figure 1b. Clearly, setting $Q(z) = 0$ reduces the parametrized predictive controller, Eq. 24, to the nominal predictive controller, Eq. 20.

Note that in contrast to the standard Youla parametrization approach, the transfer function $X(z) + M(z)Q(z)$ appears in the feedback path of Figure 1b, instead of appearing in the control block immediately preceding the plant. This deliberate departure from the standard approach, in conjunction with the factorizations (Eqs. 18 and 19) that make use of the nominal closed-loop polynomial, introduces highly desirable properties in the parametrized input-output maps as explained in the sequel.

Proposition. The nominal control loop of Figure 1a and the parametrized control loop of Figure 1b have identical servo transfer functions $y(z)/r(z)$ and $u(z)/r(z)$.

Proof. The proposition is proved trivially by carrying out block-diagram algebra on each figure to derive in both cases the servo transfer functions $y(z)/r(z)$ and $u(z)/r(z)$ that are immediately obtained after a rearrangement of factors in Eqs. 12 and 13.

Corollary. Given that the nominal controller, Eq. 5, is a predictive controller, then the parametrized controller, Eq. 24, is also a predictive controller.

Proof. If Eq. 5 is a predictive controller, then by definition it yields a control sequence $\{u(t)\}$ that minimizes the predictive performance index 4 for any prescribed setpoint trajectory $\{r(t)\}$. From Proposition 1 it follows that, for the given setpoint trajectory, the parametrized controller, Eq. 24, will also produce the same control sequence due to the equality of the servo transfer function $u(z)/r(z)$. It follows then that the parametrized controller is also a predictive controller because it yields a control sequence that minimizes Eq. 4.

Since any allowable parameter $Q(z)$ yields the same servo transfer functions $y(z)/r(z)$ and $u(z)/r(z)$, the parametrized controller has the intrinsic capability of preserving the nominal servo performance. Also note that although the terms containing $Q(z)$ effectively cancel out in the servo transfer functions, the transfer function $e(z)/w_1(z) = M(z)[Y(z) - N(z)Q(z)]$ in Figure 1b is affine in $Q(z)$, as in the standard Youla parametrization method. This allows the loop sensitivity to be shaped while simultaneously retaining nominal performance.

Design of Robust Predictive Controllers

When the nominal model, Eq. 1, is not exact due to the presence of modeling errors, the plant transfer function $g(z)$ may be written in the form

$$g(z) = g_0(z) + \Delta(z), \tag{25}$$

where $g_0(z) = B(z)/A(z)$ is the nominal plant model, and $\Delta(z)$ is an unstructured perturbation. Without loss of generality, the developments are specialized for the case of additive perturbations satisfying

$$|\Delta(e^{i\omega})| \leq |W(e^{i\omega})| \quad \forall\omega, \tag{26}$$

where the uncertainty weight $W(z)$ is a stable and proper transfer function, and the perturbation $\Delta(z)$ is assumed to be such that $g(z)$ and $g_0(z)$ have the same number of unstable poles. The case of multiplicative perturbations, as well as other typical unstructured uncertainty representations (Francis, 1987), can be treated in an analogous way. The inverse multiplicative model (Maciejowski, 1989), which allows for $g(z)$ and $g_0(z)$ to have different number of unstable poles even when the perturbation $\Delta(z)$ is stable, can be of particular relevance to chemical processing systems. The situation where the nominal model and the uncertain process have a different number of unstable poles may arise in processing systems, for example, when the process evolves through a metastable steady state in a continuous stirred tank reactor (CSTR) with multiple steady states.

The objective is to design a robust predictive controller that stabilizes the closed loop for all the members of the uncertain family of plants (Eqs. 25 and 26) and that in the nominal case, where $\Delta(z) = 0$, it recovers the performance of a nominal predictive controller (Eq. 5) that is designed solely on the basis of the nominal model $g_0(z)$. The stability robustness of the closed loop shown in Figure 2, which includes the parametrized controller, Eq. 24, and the uncertain family of plants (Eqs. 25 and 26) can be analyzed using H_∞ theory concepts as shown in Theorem 1 below.

Theorem. A necessary and sufficient condition for the robust stability of the closed-loop system of Figure 2 is the inequality condition

$$\|W(z)C(z)S_l(z)\|_\infty < 1, \tag{27}$$

where

$$C(z) := \frac{X(z) + M(z)Q(z)}{Y(z) - N(z)Q(z)} \tag{28}$$

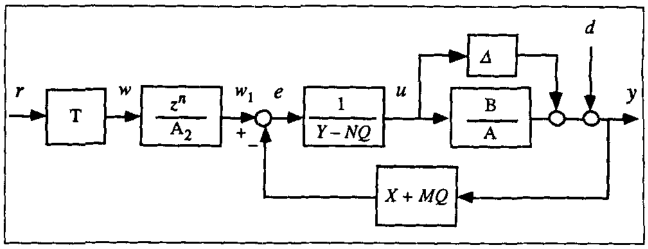

Figure 2. Structure of a robust predictive controller for a plant with an additive uncertainty.

An exogenous disturbance $d(t)$ affects the plant output.

and

$$S_l(z) := M(z)[Y(z) - N(z)Q(z)]. \tag{29}$$

Proof. Since the operators located between signals $r(z)$ and $w_1(z)$ are stable, it suffices to determine robust stability conditions for the loop obtained by ignoring the operator $z^n T(z)/A_2(z)$ and considering $w_1(z)$ as a bounded setpoint signal. The proof is then completed applying the Small-Gain theorem (Maciejowski, 1989) to the loop in question, according to the following procedure. First write $\Delta(z) := \Delta'(z)W(z)$, where $\Delta'(z)$ is a stable transfer function satisfying $\|\Delta'(z)\|_\infty \leq 1$, and define the Δ'-perturbation input variable $q(z) := W(z)u(z)$ and output variable $p(z) := \Delta'(z)q(z)$. Clearly, the Δ'-perturbation variables are related through the expression $q(z) = -W(z)C(z)S_l(z)p(z)$, where $C(z)$ and the loop sensitivity function $S_l(z)$ are as defined in Eqs. 28 and 29, respectively. Finally, the necessary and sufficient robust stability condition (27) follows by applying the Small-Gain Theorem to the transfer function $-W(z)C(z)S_l(z)$.

Robust synthesis

We propose a systematic procedure for solving the robust synthesis problem without resorting to approximations for the Youla parameter. A particular challenge to the design problem posed is the objective of including an integrator in the robustified controller in order to guarantee effective disturbance rejection. In the next subsection we develop a design technique for the base case where the plant is unstable and has no poles on the unit circle. The following subsection treats the case where the plant is unstable but has poles on the unit circle, as in the case of an integrator. The third and final subsection treats the stable plant case.

Base Case for Robust Design—Unstable Plant with no Poles on the Unit Circle. Consider the robust predictive controller design problem for the case where the nominal plant model $g_0(z)$ is unstable but has no poles on the unit circle. The synthesis problem is attacked by rewriting the robust stability condition (27) in the equivalent model-matching form

$$\|T_1(z) - T_2(z)Q(z)\|_\infty < 1, \tag{30}$$

where

$$T_1(z) = W(z)X(z)M(z) \tag{31}$$

$$T_2(z) = -W(z)M(z)^2. \tag{32}$$

Inequality 30, which is affine in the unknown Youla parameter $Q(z)$, is obtained by substituting Eqs. 28 and 29 into inequality 27. The model-matching problem is commonly approached in the context of H_∞ control theory using the γ-iteration process (Glover, 1984), where Eq. 30 is substituted by the alternative inequality

$$\|T_1(z) - T_2(z)Q(z)\|_\infty \leq \gamma, \tag{33}$$

where γ is a positive scalar parameter selected by the designer. A robust design is obtained if a Youla parameter $Q(z)$ is found for a specified $\gamma < 1$. The key is then to be able to synthesize a Youla parameter with a reliable algorithm.

One possible venue for synthesizing $Q(z)$ is to map the operators in Eq. 30 into the Laplace domain using the bilinear transformation, and then using techniques appropriate for synthesis for continuous time systems (Glover, 1984). While this procedure is sometimes useful, it suffers from a number of disadvantages. In particular, the resulting continuous-time problem may be ill-conditioned posing serious numerical difficulties. In addition, discrete poles located at zero (such as those caused by pure delays) must be excluded in order to solve the problem in continuous time, because otherwise the Glover procedure cannot guarantee a state-space solution (Rotstein and Sideris, 1992). In order to avoid the complications and subtleties associated with the bilinear mapping, we advocate the use of a z-domain technique proposed by Rotstein and Sideris for solving the model-matching problem (Rotstein and Sideris, 1992; Rotstein, 1993). Their algorithm solves the problem of approximating a stable transfer function $R(z)$ with an antistable (all poles outside the unit circle) transfer function $Q_R^{\sim}(z)$, where the tilde denotes the conjugate operation $Q_R^{\sim}(z) := Q_R(1/z)$. The problem, also known as the Nehari extension problem (Maciejowski, 1989), calls for finding an antistable function $Q_R^{\sim}(z)$ such that

$$\|R(z) - Q_R^{\sim}(z)\|_\infty \leq \gamma. \tag{34}$$

The Appendix gives relevant details of the Rotstein–Sideris algorithm, which yields the set of all the solutions to Eq. 34. The only inputs required are the transfer function $R(z)$ and the scalar γ. A necessary condition for the existence of a solution is that γ must be greater than the Hankel norm of $R(z)$, that is, $\gamma \geq \|R(z)\|_H$. One solution of particular significance is the central controller, denoted $Q_{R,c}(z)$, because the set of all the solutions can then be found as an explicit function of $Q_{R,c}(z)$.

The model-matching problem, Eq. 33, can be written in the form Eq. 34 through a series of norm-preserving transformations as detailed in the Appendix. The procedure consists of first factoring $T_2(z)$ in the form

$$T_2(z) = T_{ap}(z)T_{mp}(z), \tag{35}$$

where $T_{ap}(z)$ is an all-pass function and $T_{mp}(z)$ is a stable minimum-phase function, and then finding $T_{ap}^{\sim}(z) = T_{ap}(1/z)$ and carrying out the decomposition

$$T_{ap}^{\sim}(z)T_1(z) = R_a(z) + R_s(z), \tag{36}$$

where $R_a(z)$ and $R_s(z)$ are an antistable and a stable transfer functions, respectively. As shown in the Appendix the transfer function required for the Nehari problem, Eq. 34, is

$$R(z) = R_a^{\sim}(z). \tag{37}$$

The central solution $Q_{R,c}(z)$ is obtained from an explicit formula (see Eq. A4 in the Appendix), and a solution to the original model-matching problem, Eq. 30, is simply recovered as

$$Q(z) = T_{mp}^{-1}(z)(R_s(z) + Q_{R,c}(z)). \tag{38}$$

The final robust predictive controller design for the base case is obtained by substituting in the structure 24 the Youla parameter, Eq. 38, and the factorizations 29 and 30.

Robust Design for the Case of an Unstable Plant with Poles on the Unit Circle. When the nominal plant model $g_0(z)$ has poles on the unit circle, the standard H_∞ control theory is no longer applicable. In addition, the factorization 35 is no longer feasible because no minimum-phase stable transfer function can possibly satisfy the equality. This difficulty is circumvented by introducing a change of variable that maps unit-circle poles to a circle of larger radius. Let $z = \zeta/\rho$, where $\rho > 1$ is a scalar, and define the operators

$$T_1'(\zeta) = T_1(\zeta/\rho) \tag{39}$$

and

$$T_2'(\zeta) = T_2(\zeta/\rho). \tag{40}$$

Note that the mapping $z = \zeta/\rho$ is a special case of the well-known bilinear transformation. The effect of the transformation is to map all the z-plane poles located on the unit circle into ζ-plane poles located on a circle of radius ρ. Using Eqs. 39 and 40, the base-case design problem, Eq. 33, can be posed in terms of the transformed variable ζ to yield

$$\|T_1'(\zeta) - T_2'(\zeta)Q'(\zeta)\|_\infty < \gamma. \tag{41}$$

Problem 41 can be solved for $Q'(\zeta)$ using the base-case algorithm described in the preceding subsection. The z-domain Youla parameter is simply recovered by transforming the result back to the original space, that is,

$$Q(z) = Q'(\rho z), \tag{42}$$

and the final robust predictive controller design for this case is obtained by substituting in the structure 24 the Youla parameter, Eq. 42, and the factorizations 29 and 30.

Using the Maximum Modulus Theorem, it follows that the transformed design problem, Eq. 41, is related to the original problem, Eq. 30, through the inequality

$$\|T_1'(\zeta) - T_2'(\zeta)Q'(\zeta)\|_\infty \geq \|T_1(z) - T_2(z)Q(z)\|_\infty. \tag{43}$$

Therefore, the transformed design represents only a sufficient condition for stability. If no $Q'(\zeta)$ can be found that satisfies Eq. 41, then a smaller value for ρ is adopted and the design is repeated.

Robust Design for the Case of a Stable Plant. When the nominal plant model $g_0(z)$ is stable, the robust design is straightforward. In the normal case where the weight $W(z)$ is minimum phase (i.e., it does not have zeros on or outside the unit circle), a solution to the robust synthesis problem, Eq. 30, is obtained by setting $T_1(z)-T_2(z)Q(z)=T_3(z)$, where $T_3(z)$ is a user-specified stable biproper transfer function that satisfies the contraction condition $\|T_3(z)\|_\infty<1$. Solving for the unknown parameter as a function of $T_3(z)$ yields the design equation

$$Q(z)=T_2^{-1}(z)(T_1(z)-T_3(z)), \tag{44}$$

where $Q(z)$ is clearly a stable transfer function because $T_2(z)$ is minimum phase. In the alternative case, where the weight $W(z)$ is nonminimum phase, the design equation used is

$$Q(z)=T_2^{-1}(z)(T_1(z)-W(z)T_3(z)), \tag{45}$$

which is obtained by setting $T_1(z)-T_2(z)Q(z)=W(z)T_3(z)$, where $T_3(z)$ is a user-specified stable biproper transfer function satisfying $\|W(z)T_3(z)\|_\infty<1$. The final robust predictive controller design results by substituting in the structure 24 the Youla parameter, Eq. 44 or Eq. 45, along with the factorizations 29 and 30.

Robust design with steady-state disturbance rejection

In process-control applications the controller is often required to deliver effective disturbance-rejection performance. This section presents a method for introducing integral action in the robustified controllers, thus ensuring offset-free regulation in the presence of asymptotically constant disturbances. Given that the nominal predictive controller, Eq. 5, leads to the nominal regulation transfer function

$$\frac{y(z)}{d(z)}=\frac{\mathrm{A}(z)\mathrm{R}(z)}{\mathrm{A}(z)\mathrm{R}(z)+\mathrm{B}(z)\mathrm{S}(z)}, \tag{46}$$

it follows that $\lim_{t\to\infty} y(t)=0$ for step disturbances as well as for other disturbances with a constant steady state because $\mathrm{R}(1)=0$. On the other hand, from Figure 2 it follows that the nominal regulation transfer function for the robustified predictive controller, Eq. 25, is

$$\frac{y(z)}{d(z)}=\frac{\mathrm{A}(z)\mathrm{R}(z)}{\mathrm{A}(z)\mathrm{R}(z)+\mathrm{B}(z)\mathrm{S}(z)}-\frac{\mathrm{A}(z)\mathrm{B}(z)}{\mathrm{A}_1^2(z)}Q(z). \tag{47}$$

Because the synthesis procedures described in the previous section do not necessarily yield a Youla parameter satisfying $Q(1)=0$, the robustified predictive controller may display unacceptable nominal regulation performance at the steady state unless the nominal plant has an integrator (i.e., $\mathrm{A}(1)=0$) or is a self-regulating process ($\mathrm{B}(1)=0$). The robust predictive control design for integrating plants is carried out as indicated in the subsection on robust design for an unstable plant with poles on the unit circle, and the design for self-regulating plants is treated as in the three preceding subsections, depending on the location of the poles.

Clearly, the robust predictive controller will attain perfect steady-state disturbance rejection for all the plants belonging to the uncertain family, Eq. 25, only if the Youla parameter has a zero gain, that is, $Q(1)=0$. This gain constraint can be introduced in the robust predictive control design through a simple modification of the factorizations 29 and 30. First the integrator is extracted from the nominal predictive controller by writing $\mathrm{R}(z)=(z-1)\mathrm{R}'(z)$, and then Eq. 28 is rewritten in the form

$$\frac{\mathrm{A}(z)(z-1)\mathrm{R}'(z)}{\mathrm{A}_1(z)\mathrm{A}_2(z)}+\frac{\mathrm{B}(z)\mathrm{S}(z)}{\mathrm{A}_1(z)\mathrm{A}_2(z)}=1. \tag{48}$$

Introducing the modified coprime factorization

$$\hat{M}(z):=\frac{(z-1)\mathrm{A}(z)}{z\mathrm{A}_1(z)},\qquad \hat{N}(z):=\frac{\mathrm{B}(z)}{\mathrm{A}(z)} \tag{49}$$

and

$$\hat{X}(z):=\frac{S(z)}{\mathrm{A}_2(z)},\qquad \hat{Y}(z):=\frac{z\mathrm{R}'(z)}{\mathrm{A}_2(z)} \tag{50}$$

leads to operators that satisfy the Diophantine equation $\hat{N}(z)\hat{X}(z)+\hat{M}(z)\hat{Y}(z)=1$. The modified form of inequality 33 can then be written as

$$\|\hat{T}_1(z)-\hat{T}_2(z)\hat{Q}(z)\|_\infty<\gamma^{*} \tag{51}$$

with

$$\hat{T}_1(z)=W(z)\hat{X}(z)\hat{M}(z) \tag{52}$$

$$\hat{T}_2(z)=-W(z)\hat{M}(z)^2. \tag{53}$$

Note that the definition of $\hat{M}(z)$ in Eq. 49 is equivalent to designing a controller for a nominal plant that has been augmented by an integrator. It is now possible to proceed to solve Eq. 51 for a parameter $\hat{Q}(z)$ using the base-case algorithm, as described in the subsection on robust design for an unstable plant with poles on the unit circle. After a solution to Eq. 51 is found, the Youla parameter $Q(z)$ used in the parametrized predictive control structure of Figure 2 is constructed by reassociating the augmented-plant integrator with the controller to obtain

$$Q(z)=\frac{(z-1)}{z}\hat{Q}(z). \tag{54}$$

The final robust predictive controller design for this case is obtained by substituting in the structure 24 the Youla parameter (Eq. 54) and the factorizations (Eqs. 29 and 30). The resulting controller includes an integrator since Eq. 54 satisfies the zero-gain condition $Q(1)=0$.

Finally, it is useful to note that when the nominal model $g_0(z)$ is stable, it may be possible to include integral action in the robust predictive controller through simple specifications. In particular, this is readily accomplished by adding to the design equation (Eq. 44) the gain-equality specification $T_3(1) = T_1(1)$ if the uncertainty weight $W(z)$ is minimum phase, or adding the gain-equality constraint $T_3(1) = T_1(1)/W(1)$ to Eq. 45 if the weight is nonminimum phase. These specifications lead to the desired offset-free condition $Q(1) = 0$. However, if $T_1(1) > 1$, these simple designs are infeasible because $T_3(z)$ cannot simultaneously satisfy the contraction-mapping and the gain-equality conditions. In that case, the robust design with integral action must be done following the general procedure, Eqs. 49–54.

Example

In this section we illustrate the use of the robust design technique via an example. Note that the robust synthesis technique proposed in this article is applicable to both stable and unstable processes. In contrast, most of the alternative robust predictive control strategies documented in the literature are simply not applicable to unstable plants. In order to maximize the impact of the example we adopt an unstable nominal plant for the illustration. Even though open-loop unstable chemical processes are not predominant in the chemical industry, there is nevertheless a number of such processes that are of economic importance (Muske and Rawlings, 1993). In particular, the unstable batch chemical reactor studied by Rotstein and Lewin (1992) is characterized by a plant model with one unstable pole. The wire-catalyzed combustion reactors investigated by Sheintuch (1989) and the network of cascaded exothermic CSTRs analyzed by Georgiou et al. (1989) include plant models with two and three unstable poles, respectively. Consider the second-order nominal plant model

$$g_0(z) = \frac{z+0.2}{z^2 - 0.6z + 1.12}, \tag{55}$$

which has two complex-conjugate poles that lie outside of the unit circle. The nominal plant is subject to an additive uncertainty perturbation characterized by the weight

$$W(z) = \frac{0.63z + 0.6174}{z + 0.5}, \tag{56}$$

which reflects the extent of the model uncertainty measured in the frequency domain. Although the specification of appropriate uncertainty weights is a subject of current research, insight into practical procedures is given in a number of recent publications (see, for example, Latchman and Crisalle, 1995; Cluett et al., 1994; Kuong and MacGregor, 1993).

Three controllers are designed: (1) a nominal predictive controller (NPC); (2) a robust predictive controller (RPC); and (3) a robust predictive controller with integral action (RPCI). The nominal predictive controller is of form Eq. 5, and is realized using the design parameters $N_y = 4$, $N_u = 2$, and $\gamma = 0$, and the design equations given in the Appendix, to arrive at the polynomials

$$\mathrm{R}(z) = z^2 - 0.8039z - 0.1961$$

$$\mathrm{S}(z) = 0.8639z^2 - 1.579z + 1.0984$$

$$\mathrm{T}(z) = 0.2914z^4 - 0.0156z^3 + 0.366z^2 + 0.3243z,$$

which leads to a NPC that stabilizes the closed loop when the uncertainty is neglected. However, the NPC is not robustly stabilizing because it violates the robust stability condition 27, that is, $\|W(z)C(z)S_l(z)\|_\infty = 2.9 > 1$, where $C(z)$ and $S_l(z)$ are calculated using $Q(z) = 0$ in Eqs. 28 and 29. This implies that the nominal predictive controller will fail to stabilize the closed loop for some plants belonging to the family of uncertain plants. Efforts to detune the nominal controller by extending the prediction horizon up to the value $N_y = 30$, as well as increasing the control horizon to values greater than the number of unstable poles, also fail to yield robustly stabilizing controllers. Clearly, the lack of success of the detuned controllers in robustly stabilizing the feedback loop is due to the fact that the detuning procedure is conceived without taking into account the explicit additive uncertainty model available through the description (Eq. 56).

The RPC design is of the form (Eq. 24). Since the unstable nominal plant has no poles on the unit circle, the design proceeds as discussed in the base case (see the subsection on the base case for robust design). The transfer functions $T_1(z)$ and $T_2(z)$ are formed as prescribed in Eqs. 31 and 32, using $\mathrm{A}_2(z) = z^2$. To solve the Nehari extension problem we use $\gamma = 0.99$, which is acceptable since it exceeds the limiting Hankel norm value $\|R(z)\|_H$ and is less than one as required by Eq. 30. The central controller found using the base-case algorithm leads to a Youla parameter $Q(z) = \mathrm{N}_Q(z)/\mathrm{D}_Q(z)$ of order 8, with

$$\mathrm{N}_Q(z) = -0.8688z^8 + 2.026z^7 - 2.699z^6 + 1.515z^5 + 0.4916z^4 - 2.267z^3 + 2.144z^2 - 1.297z + 0.3668$$

$$D_Q(z) = z^8 - 0.6525z^7 + 1.958z^6 + 0.4477z^5 + 0.1928z^4 + 1.809z^3 - 0.5414z^2 + 0.926z.$$

The RPC transfer functions $Y(z) - N(z)Q(z)$ and $X(z) + M(z)Q(z)$ are of order 9 in their minimal forms. The controller is robustly stabilizing because $\|W(z)C(z)S_l(z)\|_\infty = 0.35 < 1$.

Finally, the design of the RPCI is carried out as indicated in the section on robust design with steady-state disturbance rejection, using again the specification $\gamma = 0.99$. Since this control design includes an integrator, the bilinear transformation $z = \zeta/\rho$ discussed in the subsection on robust design for an unstable plant with poles on the unit circle is implemented using the value $\rho = 1.1$. The procedure leads to a Youla parameter $Q(z) = (z-1)\mathrm{N}_Q(z)/(z\mathrm{D}_Q(z))$ of order 9, with

$$\mathrm{N}_Q(z) = -0.8648z^8 + 2.534z^7 - 2.766z^6 + 0.9587z^5 + 0.6691z^4 - 1.006z^3 + 0.6545z^2 - 0.2434z + 0.0389$$

$$D_Q(z) = z^8 - 2.19z^7 + 2.181z^6 - 1.129z^5 - 0.2604z^4 + 0.8927z^3 - 0.7056z^2 + 0.3307z - 0.0734.$$

The resulting RPCI transfer functions $Y(z) - N(z)Q(z)$ and $X(z) + M(z)Q(z)$ are of order 10 in their minimal forms, and $Q(1) = 0$, as desired. The RPCI controller is robustly stabilizing because $\|W(z)C(z)S_l(z)\|_\infty = 0.49 < 1$.

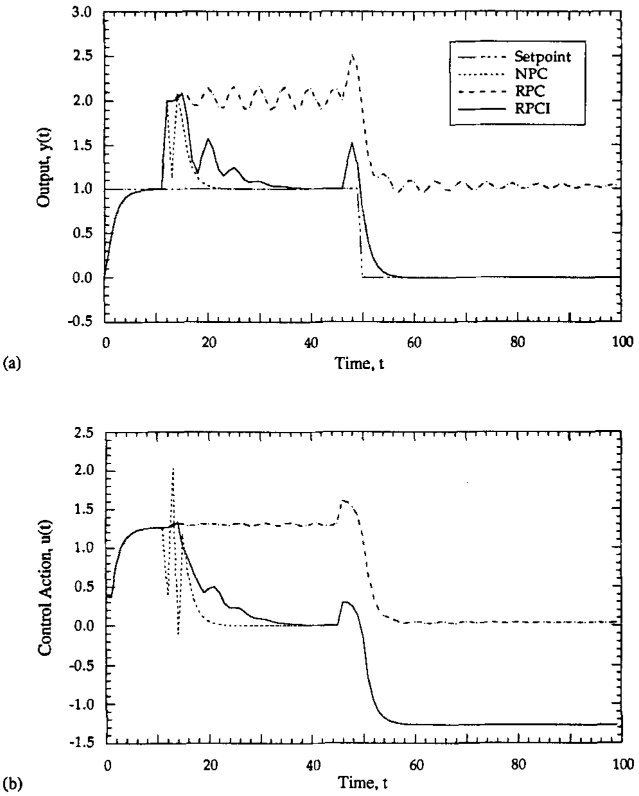

Figure 3. Comparison of the performance of the NPC, the RPC, and the RPCI designed in the example for a plant with no uncertainty.

The setpoint $r(t)$ experiences changes at instants $t = 0$ and $t = 50$, and a unit-step disturbance $d(t)$ (not shown) is introduced at $t = 12$.

Figure 3 shows the results of a closed-loop simulation test carried out to evaluate the nominal servo and regulation performances of the three control designs. The process is assumed to match exactly with the nominal model, that is, $\Delta(z) = 0$. The servo test features step changes in the setpoint $r(t)$ at instants $t = 0$ and $t = 50$, and the regulation test is carried out by introducing a unit-step disturbance $d(t)$ (not shown in the figure) at $t = 12$.

Figure 3a shows that during the first 12 samples, where $d(t) = 0$, all three controllers display identical dynamics. This is the expected result, since Proposition 1 guarantees that the nominal predictive controller and the robustified controllers have identical servo transfer functions, independent of the value of $Q(z)$. Also as expected, the control-output trajectories shown in Figure 3b are also identical during this interval.

The three controllers differ, however, in their regulation behavior. When the disturbance is introduced at $t = 12$, the NPC rejects the disturbance effectively, quickly returning the output to the setpoint, as shown in Figure 3a. In marked contrast, the RPC fails to reject the effect of the disturbance, and displays steady-state offset. The RPCI, on the other hand, succeeds in rejecting the disturbance, albeit with slower dynamics than the nominal controller. Figure 3b shows that the NPC achieves the disturbance rejection at the expense of fairly energetic control actions that follow the onset of the disturbance. On the other hand, the RPCI prescribes more conservative input adjustments, typical of robust controllers. In many practical situations, the smoother dynamics of the RPCI design may be highly preferable over the more aggressive behavior of the NPC.

As a final remark, note that all the controllers anticipate the occurrence of setpoint change at $t = 50$, as evidenced by the early adjustments in control action that take place starting at instant $t = 46$, as shown in Figure 3b. This anticipatory behavior is a characteristic of predictive controllers. Since the prediction horizon N_y has been selected equal to four samples, the controllers naturally initiate adjustments at instant $t = 46$ when the prediction horizon first permits detection of the upcoming setpoint change. This observation verifies that the robustified control designs can legitimately be classified as predictive controllers, as claimed in Corollary 1.

Figure 4 shows the results of a closed-loop simulation test for a perturbed plant ($\Delta(z) \neq 0$). As in the previous example, the setpoint changes at instants $t = 0$ and $t = 50$, and an external unit-step disturbance $d(t)$ is introduced at $t = 12$. The figure shows that the NPC is unable to control the plant, causing unstable closed-loop dynamics. In marked contrast, both robust predictive control designs RPC and RPCI have stable responses. As expected, the RPC controller suffers from steady-state offset due to the lack of an integrator, whereas the RPCI controller has offset-free behavior and manages to reject the disturbance without excessive control action.

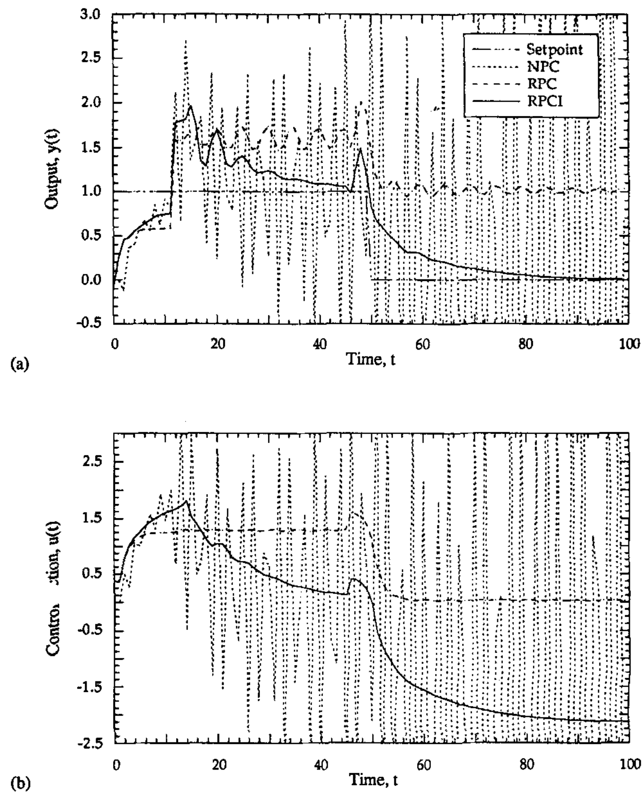

Figure 4. Comparison of the performance of the NPC, the RPC, and the RPCI, designed in the example for a perturbed plant belonging to the uncertainty description.

The setpoint $r(t)$ changes at instants $t = 0$ and $t = 50$, and a unit-step disturbance $d(t)$ (not shown) is introduced at $t = 12$.

Conclusions

A systematic method for robustifying unconstrained predictive controllers has been proposed. The technique succeeds in preserving nominal servo performance due to the unconventional feedback configuration adopted for the parametrized controller, and also due to a coprime factorization that makes use of the characteristic polynomial of the nominal closed loop. A significant feature of the proposed method is its applicability to both stable and unstable plants. Another advantage of the robust synthesis technique is that it permits the incorporation of integral action in the robustified controllers, making the resulting controllers more useful for practical applications. Although the technique is developed for single-input, single-output systems, we anticipate that the methodology of this article extends naturally to the multivariable case. A limitation of the proposed method is that it does not take into account the effect of input saturation. A possible venue to address this problem is to restate the H_∞ problem, Eq. 33, as a mixed l_1/H_∞ problem that explicitly takes into consideration constraints on the manipulated variable. Research is currently in progress addressing the robustification of constrained predictive controllers.

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Appendix

Rotstein-Sideris solution to the model-matching problem

The transformation of the model-matching problem (Eq. 33) to the form (Eq. 34) can be done in a straightforward fashion as follows. First $T_2(z)$ must be factored as shown in Eq. 35. Then, using the property that $\|T_{ap}^{\sim}(z)G(z)\|_\infty = \|G(z)\|_\infty$, it is possible to write the equalities

$$\|T_1(z) - T_2(z)Q(z)\|_\infty = \|T_{ap}^{\sim}(z)T_1(z) - T_{mp}(z)Q(z)\|_\infty$$
$$= \|R_a(z) + R_s(z) - T_{mp}(z)Q(z)\|_\infty. \quad (A1)$$

Defining $R^{\sim}(z) := R_a(z)$ and $Q_R(z) := R_s(z) - T_{mp}(z)Q(z)$, and using the property $\|G^{\sim}(z)\|_\infty = \|G(z)\|_\infty$, the previous equality readily reduces to the desired form

$$\|T_1(z) - T_2(z)Q(z)\|_\infty = \|R(z) - Q_R^{\sim}(z)\|_\infty. \qquad \text{(A2)}$$

Note that the factorization (Eq. 35) is not possible for nominal plants with poles on the unit circle. Such special cases must be treated as discussed in the section on robust design for an unstable plant with poles on the unit circle. A solution to the model-matching problem (Eq. 34) is proposed in (Rotstein and Sideris, 1992). The main result is stated in the theorem below.

Theorem. Let $R(z)$ be a stable and proper transfer function with a minimal state-space realization

$$R(z) := C(zI - A)^{-1}B + D = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

and let the scalar $\gamma > \|R(z)\|_H$. Then the set of all antistable proper transfer functions $Q^{\sim}(z)$ satisfying $\|R - Q_R^{\sim}\|_\infty \leq \gamma$ are given by the conjugate of

$$Q_R(z) = F_l(Q_{R,c}(z), \Phi(z)), \qquad \text{(A3)}$$

where the operator $F_l(\cdot,\cdot)$ represents a linear-fractional transformation, and $\Phi(z)$ is a stable and proper transfer function satisfying $\|\Phi(z)\|_\infty < \gamma$. Transfer function $Q_{R,c}(z)$ is the central solution defined as

$$Q_{R,c}(z) = \left(\begin{array}{c|cc} A - B(B^TL_oA + E_{11}^TC)N & (AL_cC^T + BE_{11}^T) & -BD_{21}^T \\ \hline (E_{11}^TC + B^TL_oA)N & D_{11}^T & D_{21}^T \\ -D_{12}^TCN & D_{12}^T & D_{22}^T \end{array}\right) \qquad \text{(A4)}$$

where L_c and L_o denote the controllability and observability grammians, respectively, of $R(z)$, which are the solutions to the discrete-time Lyapunov equations

$$L_c = AL_cA^T + BB^T \qquad \text{(A5)}$$

$$L_o = A^TL_oA + C^TC \qquad \text{(A6)}$$

and $N = (\gamma^2 I - L_cL_o)^{-1}$. Rotstein (1993) give details on the construction of auxiliary matrices E_{11} and D_{ij}, which depend on matrices A, B, C, D, L_c, and L_o, and on the value of γ.

The central controller, Eq. A4, can be directly calculated from a solution to Eqs. A5 and A6, and the appropriate definitions for matrices E_{11} and D_{ij}. The use of the Hankel-norm condition $\gamma > \|R(z)\|_H$ stems from the fact that the minimum value of $\|R - Q_R^{\sim}\|_\infty$ over the stable proper transfer functions $Q(z)$ is exactly equal to $\|R(z)\|_H$. Hence, the only legitimate values of γ are those exceeding the Hankel norm.

Design equations for nominal predictive control

This section provides specific design equations used to synthesize a nominal predictive controller following the approach of Crisalle et al. (1989). An equivalent formulation is given in McIntosh et al. (1991). The final design equations for the polynomials (Eqs. 6–8) that appear in the predictive control law (Eq. 5) are

$$R(z) = z^n\left[1 + z^{-1}\sum_{i=1}^{N} k_i\Gamma_i(z^{-1})\right] \qquad \text{(A7)}$$

$$S(z) = z^n\left[\sum_{i=1}^{N} k_iF_i(z^{-1})\right] \qquad \text{(A8)}$$

$$T(z) = \sum_{i=1}^{N} k_iz^i, \qquad \text{(A9)}$$

where the design operators $F_i(z^{-1})$ and $\Gamma_i(z^{-1})$, and the coefficients k_i, $i = 1, 2, \ldots, N$ are determined from the process model according the following procedure. First rewrite the nominal plant model, Eqs. 1–3, in the equivalent form

$$A_1(z^{-1})y(z) = z^{-1}B_1(z^{-1})u(z) \qquad \text{(A10)}$$

involving inverse powers of z, where $A_1(z^{-1})$ and $B_1(z^{-1})$ are related to Eqs. 2 and 3 in an obvious manner and are of the form

$$A_1(z^{-1}) = 1 + a_{1,1}z^{-1} + a_{1,2}z^{-2} + \cdots + a_{1,n_a}z^{-n_a} \qquad \text{(A11)}$$

$$B_1(z^{-1}) = b_{1,0} + b_{1,1}z^{-1} + \cdots + b_{1,n_b}z^{-n_b}. \qquad \text{(A12)}$$

To obtain the design operators $F_i(z^{-1})$, which are polynomials of degree n (the order of the plant), solve the set of Diophantine equations

$$E_i(z^{-1})\Delta(z^{-1})A_1(z^{-1}) + z^{-i}F_i(z^{-1}) = 1, \qquad i = 1, 2, \ldots, N, \qquad \text{(A13)}$$

which also yields the intermediate polynomials $E_i(z^{-1})$ of degree $i-1$. The second design operators, the polynomials $\Gamma_i(z^{-1})$ of degree n, are obtained by decomposing the product $E_i(z^{-1})B(z^{-1})$ in the form

$$E_i(z^{-1})B_1(z^{-1}) = G_i(z^{-1}) + z^{-i}\Gamma_i(z^{-1}), \qquad \text{(A14)}$$

where polynomials $G_i(z^{-1})$ of degree $i-1$ are known as the *dynamic polynomials*, and are characterized by the fact that their coefficients are the sampled values of the step response of the plant (Eq. A10). In turn, the coefficients of the dynamic polynomials are used to define the nonzero elements of the Toeplitz matrix G_{N_u} known as the *truncated dynamic matrix*, which contains only N_u columns. Finally, the coefficients k_i, $i = 1, 2, \ldots, N$ are obtained as the components of the gain vector $k^T = [k_1\ k_2,\ \ldots,\ k_N]$, calculated from the expression

$$k^T = [1\quad 0\quad \cdots\quad 0]\left(G_{N_u}^TG_{N_u} + \lambda I\right)^{-1}G_{N_u}^T. \qquad \text{(A15)}$$

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